Recitation 7. April 27

Focus: algebraic & geometric multiplicity, Jordan normal form, complex eigenvalues & eigenvectors, symmetric matrices

Given an $n \times n$ matrix A and an eigenvalue λ, then its:

$$\boxed{\textbf{algebraic multiplicity}} = \text{multiplicity of } \lambda \text{ as a root of the characteristic polynomial}$$

$$\boxed{\textbf{geometric multiplicity}} = \text{dimension of } N(A - \lambda I)$$

In general, we have the following inequality for all eigenvalues of A:

$$\boxed{\text{algebraic multiplicity} \geq \text{geometric multiplicity}}$$

If the inequality above is an equality for all eigenvalues λ of A, then $\mathbb{R}^n$ has a basis consisting only of eigenvectors, hence we can diagonalize the matrix A:

$$\boxed{A = VDV^{-1}}$$

where V is the matrix whose columns are eigenvectors, and D is the diagonal matrix with the eigenvalues on the diagonal. Even if the matrix A is not diagonalizable, we can always write it as:

$$A = V \left[\begin{array}{c|c|c|c} J_1 & 0 & \dots & 0 \\ \hline 0 & J_2 & \dots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \dots & J_t \end{array}\right] V^{-1}$$

where each J_i is a $\boxed{\textbf{Jordan block}}$ of the form:

$$\begin{bmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{bmatrix}$$

As the eigenvalues are roots of the characteristic polynomial, quite often it turns out that they are $\boxed{\textbf{complex numbers}}$:

$$\boxed{z = a + bi}$$

where $a, b \in \mathbb{R}$ and i is a symbol that satisfies the relation $i^2 = -1$. Recall the following notions:

$$\boxed{\textbf{complex conjugate}} : \overline{a + bi} = a - bi$$

$$\boxed{\textbf{absolute value}} : |a + bi| = \sqrt{a^2 + b^2}$$

You may go back and forth between $\boxed{\textbf{Cartesian coordinates}}$ a, b and $\boxed{\textbf{polar coordinates}}$ r, θ:

$$\begin{cases} r = \sqrt{a^2 + b^2} \\ \theta = \arccos\left(\frac{a}{\sqrt{a^2+b^2}}\right) \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} a = r\cos\theta \\ b = r\sin\theta \end{cases}$$

So with this in mind, we get the formula:

$$\boxed{a + bi = r(\cos\theta + i\sin\theta) = re^{i\theta}}$$

Symmetric $n \times n$ matrices are always diagonalizable with real eigenvalues and orthonormal eigenvectors:

$$\boxed{S = QDQ^{-1} = QDQ^T}$$

1. Which of the following matrices are diagonalizable? What are the algebraic and geometric multiplicities of the eigenvalues?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Solution:

2. Consider the matrix:

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

and calculate $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$.

Solution:

3. Consider the matrix:

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find its eigenvalues and eigenvectors. How many eigenvalues are complex? Are there complex conjugate pairs?

Solution:

4. Consider the matrix:

$$Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

- Can you find an easy eigenvalue without computing the characteristic polynomial?
- Compute all eigenvectors for the above easy eigenvalue
- Can you use this to determine the remaining eigenvector and eigenvalue?

Solution: